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1979 J. Phys. A: Math. Gen. 12 1341

(http://iopscience.iop.org/0305-4470/12/9/005)

J. Phys. A: Math. Gen., Vol. 12, No. 9, 1979. Printed in Great Britain

Invariant tensors for some maximal subgroups of the conformal group of space–time

J Beckers, M Jaminon and J Serpe
Université de Liège, Physique théorique et mathématique, Institut de Physique au Sart Tilman, Bâtiment B5, B4000 Liège 1, Belgium

Received 24 July 1978, in final form 15 January 1979

Abstract. Tensor fields invariant under some maximal subgroups of the conformal group of space–time are determined. The infinitesimal method is applied from conformal *point* transformations discussed by Fulton, Rohrlich and Witten. 'Electromagnetic' skew-symmetric tensors $F(x)$, 'electromagnetic' four-potentials $A(x)$ and scalar densities $\Phi(x)$ are studied with respect to the conditions of invariance under $SO(3)\otimes SO(2,1)$, $O(3,2)$ and $O(2)\otimes O(4)$.

1. Introduction

The growing interest in conformal invariance in very recent studies has been manifest since the late sixties (Barut and Brittin 1971). The conformal group of space–time is indeed one of the current main symmetry groups under study in theoretical physics. General relativity and cosmological models, quantum mechanics of the hydrogen atom, particle physics, etc, are some of the many fields which underline the need for some invariance under conformal transformations.

In fact, the conformal group of space–time has been recognised for a long time as the invariance group of Maxwell's equations (Bateman 1910, Cunningham 1910) and of the other *massless* particle equations (Dirac 1936, Kastrup 1962 and references therein). Moreover particles with *non-zero mass* and their associated fields and equations are currently studied in connection with conformal invariance (Fubini 1976, De Alfaro and Furlan 1976, De Alfaro *et al* 1976a).

General information on the conformal group can easily be found (Kastrup 1962, Mack and Salam 1969, Ferrara *et al* 1973). Furthermore, a recent subalgebra analysis of the conformal algebra has been achieved (Patera *et al* 1974, 1977a). In particular, the *maximal* subgroups of the conformal group are known through this subalgebra analysis (Patera *et al* 1977b) and some of these maximal subgroups actually play a prominent role in the *physical* studies above. For instance, the de Sitter groups $O(4,1)$ and $O(3,2)$ are of very special interest, as everybody knows, but, in connection with Fubini's works, the $O(3,2)$ maximal subgroup of the conformal group is exploited more extensively (Fubini 1976, De Alfaro *et al* 1976a, b, Girardello and Pallua 1977). Other recent approaches have also pointed out the particular interest of some (other) maximal subgroups like $O(2)\otimes O(4)$ and $SO(3)\otimes SO(2,1)$ (Englefield 1972, De Alfaro *et al* 1976b, 1977, Schechter 1978, Rebbi 1978, Harnad and Vinet 1978).

In this paper, we want to determine tensor fields invariant under each of these three maximal subgroups $SO(3)\otimes SO(2,1)$, $O(3,2)$ and $O(2)\otimes O(4)$ of the conformal group. From the conformal *point* transformations discussed by Fulton *et al* (1962), we shall develop an infinitesimal method in order to get the *invariance conditions* on skew-symmetric tensor fields of order two (F), four-vectors (A) and scalar densities (Φ). In a sense, this method generalises to the conformal group some recent studies on the Poincaré group and its subgroups (Bacry *et al* 1970, Combe and Sorba 1975, Beckers and Comté 1976, 1979). The method has already been mentioned elsewhere (Beckers *et al* 1978) but here we start from the conformal point transformations; we also treat applications which are not considered in the reference above.

In fact, F- and A-invariant fields will be studied because these tensors correspond to meaningful 'electromagnetic' quantities when the Maxwell theory is considered. Moreover, as scalar densities Φ enter, for instance, in up-to-date Lagrangian approaches (Fubini 1976), they will also be determined. Finally, through the proposed applications, we shall illustrate different kinds of difficulties characteristic of the infinitesimal method: the determination of the interesting $O(2)\otimes O(4)$-invariant A field will be by itself a typical example.

Section 2 will be devoted to some notations and elements on the conformal group and its maximal subgroups. Following the developments of Fulton *et al* (1962), we shall give the general invariance conditions on F, A and Φ in § 3. Section 4 will contain the specifications of the infinitesimal method and the corresponding conditions in terms of meaningful physical quantities. In § 5, we shall be concerned with some applications of the invariance conditions to the three maximal subgroups above. Section 6 will give a summary of the results and allow us to present some conclusions connected with other approaches and with differential geometry underlying an important part of such developments. In the appendix we shall discuss specific points necessary to the determination of the $O(2)\otimes O(4)$-invariant 'four-potential'.

2. Notations and comments on the maximal subgroups of the conformal group

If we refer to space–time events $x\equiv\{x^{\mu}, \mu=0,1,2,3)\}=\{x^0=t, x'=x, x^2=y, x^3=z\}$ $=\{t,\boldsymbol{r}\}$ with the (Minkowski) metric tensor $G_{\mathrm{M}}\equiv\{g^{\mu\nu}\}=\mathrm{diag}(1,-1,-1,-1)$, we have

$$x\,.\,x=(x)^2=(x^0)^2-(\boldsymbol{x})^2=t^2-r^2 \tag{2.1}$$

where

$$r=|\boldsymbol{r}|=(x^2+y^2+z^2)^{1/2}. \tag{2.2}$$

The conformal group of *space–time* is then a fifteen-parameter Lie group of (continuous) transformations corresponding to *four* translations (a^{μ}, P^{μ}), *six* restricted (proper and orthochronous) homogeneous Lorentz transformations $(\omega^{\mu\nu}=-\omega^{\nu\mu}, M^{\mu\nu})$, *one* dilatation (ρ, D) and *four* special conformal transformations (c^{μ}, C^{μ}) where $(\ldots,\ldots)$ corresponds to the associated parameters and generators respectively (Kastrup 1962, Beckers and Jaminon 1978). The corresponding *infinitesimal coordinate* transformations have the following forms:

$$x\xrightarrow{a}x':x'^{\mu}=x^{\mu}+a^{\mu}\qquad(\mu=0,1,2,3) \tag{2.3a}$$

$$x\xrightarrow{\omega}x':x'^{\mu}=x^{\mu}+\omega^{\mu}{}_{\nu}x^{\nu} \tag{2.3b}$$

$$x \xrightarrow{\rho} x' : x'^{\mu} = x^{\mu} + \rho x^{\mu} \tag{2.3c}$$

$$x \xrightarrow{c} x' : x'^{\mu} = x^{\mu} + 2x^{\mu}(c\,.\,x) - c^{\mu}(x)^2 \tag{2.3d}$$

when only first-order terms in the parameters are retained.

It is well known (Kastrup 1962, Mack and Salam 1969) that this conformal group has an O(4, 2) structure. Its subgroup analysis can evidently be determined through the subalgebra study of the O(4, 2) algebra. This has been achieved in very recent works (Patera *et al* 1977a, b, Burdet *et al* 1978) and, in particular, all the *maximal* subalgebras of the conformal algebra are known when conjugacy classes are studied under conformal transformations (Patera *et al* 1977b) *or* when conjugacy classes are studied under Poincaré transformations (Beckers *et al* 1978). *Under conformal transformations,* there are nine O(4, 2) conjugacy classes of maximal subalgebras corresponding to the following types of Lie groups: SIM(3, 1), OPT(3, 1), O(3, 2), O(4, 1), S[U(1)⊗U(2, 1)], O(2)⊗O(4), O(2)⊗O(2, 2), SO(3)⊗SO(2, 1) and SO(2, 1)⊗SO(2, 1). We shall refer to these subgroups as 'algebraically maximal' subgroups of the conformal group because their Lie algebras are maximal subalgebras of the conformal algebra. So, for brevity, we call the nine subgroups above 'the maximal subgroups of the conformal group'.

As already discussed in the introduction, we restrict ourselves to the consideration of *only three* of the maximal subgroups, i.e.:

(*a*) $\mathrm{SO}(3)\otimes\mathrm{SO}(2,1)$: $n = 6, \{\boldsymbol{J}, P^0, D, C^0\}$ (2.4)

(*b*) $\mathrm{O}(3,2)$: $n = 10, \{\boldsymbol{J}, \boldsymbol{K}, P^{\mu} + C^{\mu}\}$ (2.5)

(*c*) $\mathrm{O}(2)\otimes\mathrm{O}(4)$: $n = 7, \{\boldsymbol{J}, P^0 + C^0, \boldsymbol{P} - \boldsymbol{C}\}$ (2.6)

where n is the order of the corresponding subgroup and $\{\ldots\}$ is its corresponding set of generators (a basis for its Lie algebra) where we have introduced the usual notations $\{M^{\mu\nu}\} \equiv \{\boldsymbol{J}, \boldsymbol{K}\}$, where $\boldsymbol{J}$ and $\boldsymbol{K}$ refer to spatial rotations and to Lorentz boosts respectively.

3. Conformal point transformations and general invariance conditions

Fulton *et al* (1962, hereafter referred to as FRW) have discussed the *conformal point* transformations and their results have recently been applied (Beckers and Jaminon 1978) to covariant symmetric tensors of degree two. For brevity, we refer the reader to the third section of these two papers for further details. Here, we mention only the *invariance conditions,* under conformal point transformations, on three specific tensors of very special interest in physics:

(*a*) a skew-symmetric tensor of degree two ($F \equiv \{F_{\mu\nu}(x)\}$),
(*b*) a vector field ($A \equiv \{A_{\mu}(x)\}$) and
(*c*) a scalar density ($\Phi(x)$).

It can be shown from FRW's developments that

(*a*) $$F_{\mu\nu}(\bar{x}) = \bar{\partial}_{\mu} x^{\alpha} \bar{\partial}_{\nu} x^{\beta} F_{\alpha\beta}(x) \qquad \bar{\partial}_{\mu} \equiv \partial/\partial \bar{x}_{\mu} \tag{3.1}$$

and

$$F^{\mu\nu}(\bar{x}) = \sigma^{-2}(x) \partial_{\alpha} \bar{x}^{\mu} \partial_{\beta} \bar{x}^{\nu} F^{\alpha\beta}(x) \tag{3.2}$$

where the scalar function $\sigma(x)$ is given by

$$\sigma(x) = \tfrac{1}{4}\partial^\alpha \bar{x}_\beta \partial_\alpha \bar{x}^\beta; \tag{3.3}$$

(*b*)
$$A_\mu(\bar{x}) = \bar{\partial}_\mu x^\alpha A_\alpha(x) \tag{3.4}$$

and

$$A^\mu(\bar{x}) = \sigma^{-1}(x)\partial_\alpha \bar{x}^\mu A^\alpha(x); \tag{3.5}$$

(c)
$$\Phi(\bar{x}) = \sigma^{-1/2}(x)\Phi(x). \tag{3.6}$$

The equations (3.2), (3.5) and (3.6) will be fundamental in our study: they correspond to the general conditions of invariance under conformal point transformations on contravariant components of the F- and A-tensor fields and on an invariant scalar density Φ respectively.

4. The infinitesimal method and the corresponding invariance conditions on *F*, *A* and Φ

Consider the *point* transformation giving the components of a point $\bar{x}$ in a coordinate system S when the components of the point x are known in the *same* coordinate system. It is written

$$\bar{x}^\mu = f(x^\mu) \tag{4.1}$$

or, if we study *infinitesimal conformal* point transformations

$$\bar{x}^\mu = x^\mu + \xi^\mu \tag{4.2}$$

$$= x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu + \rho x^\mu + 2x^\mu(c\,.\,x) - c^\mu x^2 \tag{4.3}$$

where a^μ, $\omega^{\mu\nu}$, ρ and c^μ are small parameters defined in the specific transformations (2.3*a*)–(2.3*d*) belonging to the conformal group.

Through equation (3.3) we easily obtain

$$\sigma(x) = 1 + 2\rho + 4(c\,.\,x) \tag{4.4}$$

$$\sigma^{-1/2}(x) = 1 - \rho - 2(c\,.\,x) \tag{4.5}$$

$$\sigma^{-1}(x) = 1 - 2\rho - 4(c\,.\,x) \tag{4.6}$$

$$\sigma^{-2}(x) = 1 - 4\rho - 8(c\,.\,x) \tag{4.7}$$

when only first-order terms in the parameters are retained. The invariance conditions—equations (3.2), (3.5) and (3.6)—can then be developed using (4.3) and (4.5)–(4.7).

(*a*) Let us study equation (3.2) concerning the *contravariant* components of the *skew-symmetric* tensor of degree two. If we expand the left-hand side of (3.2) by a first-order Taylor development in the neighbourhood of x and its right-hand side by introducing (4.3) and (4.7), we finally get the *invariance condition on F*:

$$\mathscr{D}F(x) + \omega \wedge F(x) + 2(x \times c) \wedge F(x) - [\rho + 2c\,.\,x][x\,.\,\nabla + 2]F(x) + x^2(c\,.\,\nabla)F(x) = 0 \tag{4.8}$$

where

$$\mathscr{D} \equiv x\,.\,\omega\,.\,\nabla - a\,.\,\nabla = x_\rho \omega^{\rho\lambda}\partial_\lambda - a^\lambda \partial_\lambda \tag{4.9}$$

$$\{\nabla_\mu\} \equiv \{\partial_\mu\} = \{\partial/\partial t, \partial/\partial \boldsymbol{r}\} \tag{4.10}$$

$$(\omega \wedge F)^{\mu\nu} = \omega^{\mu}{}_{\sigma}F^{\sigma\nu} - \omega^{\nu}{}_{\sigma}F^{\sigma\mu} \tag{4.11}$$

$$[x \times c]^{\rho}{}_{\sigma} = x^{\rho}c_{\sigma} - x_{\sigma}c^{\rho} \tag{4.12}$$

and

$$[(x \times c) \wedge F]^{\mu\nu} = (x \times c)^{\mu}{}_{\sigma}F^{\sigma\nu} - (x \times c)^{\nu}{}_{\sigma}F^{\sigma\mu}. \tag{4.13}$$

(*b*) Equation (3.5) can also be developed through the above steps. With equations (4.3), (4.6), (4.9) and (4.12), we easily get the *invariance condition on* A:

$$\mathscr{D}A(x) + [\omega + 2(x \times c)] \,.\, A(x) - [\rho + 2c \,.\, x][1 + x \,.\, \nabla]A(x) + x^2(c \,.\, \nabla)A(x) = 0. \tag{4.14}$$

(c) If we consider equation (3.6), the *invariance condition on* Φ becomes with (4.3), (4.5) and (4.9):

$$\mathscr{D}\Phi(x) - [\rho + 2c \,.\, x][1 + x \,.\, \nabla]\Phi(x) + x^2(c \,.\, \nabla)\Phi(x) = 0. \tag{4.15}$$

Before we exploit the conditions (4.8), (4.14) and (4.15) in some specific cases, let us express the F and A fields in terms of more directly physical quantities so that if the Maxwell theory is concerned we could simply interpret these fields as the 'electromagnetic' tensor and four-vector. This is clear through the correspondences ($\epsilon^{123} = 1$)

$$F^{0i} = E^i \qquad \tfrac{1}{2}\epsilon^{ijk}F_{jk} = B^i \qquad (i, j, k = 1, 2, 3) \rightrightarrows F \equiv \{F^{\mu\nu}\} \equiv (\boldsymbol{E}, \boldsymbol{B}) \tag{4.16}$$

$$A^0 = V \qquad A^i = A^i \qquad (i = 1, 2, 3) \rightrightarrows A \equiv (V, \boldsymbol{A}), \tag{4.17}$$

in such a way that we can speak of the 'electromagnetic' bivector or tensor $F \equiv (\boldsymbol{E}, \boldsymbol{B})$ and of the 'electromagnetic' four-potential A.

If we define, as usual, the parametrisation of Lorentz transformations by

$$\phi^i = \omega^{0i} \qquad \theta^i = \tfrac{1}{2}\epsilon^{ijk}\omega_{jk} \rightrightarrows \{\omega^{\mu\nu}\} \equiv (\boldsymbol{\phi}, \boldsymbol{\theta}), \tag{4.18}$$

it is easy to see that

$$\omega \wedge F \equiv (\boldsymbol{\phi} \wedge \boldsymbol{B} + \boldsymbol{\theta} \wedge \boldsymbol{E}, \boldsymbol{\theta} \wedge \boldsymbol{B} - \boldsymbol{\phi} \wedge \boldsymbol{E})$$

and (4.19)

$$\omega \,.\, A \equiv (-\boldsymbol{\phi} \cdot \boldsymbol{A}, -V\boldsymbol{\phi} + \boldsymbol{\theta} \wedge \boldsymbol{A}).$$

The use of the parametrisation (4.18) leads to the differential operator (4.9):

$$\mathscr{D} \equiv (\boldsymbol{r} \cdot \boldsymbol{\phi})(\partial/\partial t) + (t\boldsymbol{\phi} + \boldsymbol{r} \wedge \boldsymbol{\theta}) \cdot (\partial/\partial \boldsymbol{r}) - a \,.\, \nabla. \tag{4.20}$$

Then, with the definitions (4.16), (4.18) and (4.20), the $(0i)$ and (ij) components of the conditions (4.8) become

$$\begin{aligned}\mathscr{D}\boldsymbol{E} + \boldsymbol{\phi} \wedge \boldsymbol{B} + \boldsymbol{\theta} \wedge \boldsymbol{E} + 2t(\boldsymbol{c} \wedge \boldsymbol{B}) - 2c^0(\boldsymbol{r} \wedge \boldsymbol{B}) + 2(\boldsymbol{r} \wedge \boldsymbol{c}) \wedge \boldsymbol{E}& \\ - [\rho + 2c \,.\, x][x \,.\, \nabla + 2]\boldsymbol{E} + x^2(c \,.\, \nabla)\boldsymbol{E} = 0&\end{aligned} \tag{4.21a}$$

and

$$\begin{aligned}\mathscr{D}\boldsymbol{B} + \boldsymbol{\theta} \wedge \boldsymbol{B} - \boldsymbol{\phi} \wedge \boldsymbol{E} - 2t(\boldsymbol{c} \wedge \boldsymbol{E}) + 2c^0(\boldsymbol{r} \wedge \boldsymbol{E}) + 2(\boldsymbol{r} \wedge \boldsymbol{c}) \wedge \boldsymbol{B}& \\ - [\rho + 2c \,.\, x][x \,.\, \nabla + 2]\boldsymbol{B} + x^2(c \,.\, \nabla)\boldsymbol{B} = 0.&\end{aligned} \tag{4.21b}$$

These relations (4.21a) and (4.21b) are the necessary and sufficient conditions of invariance of a general 'electromagnetic' tensor under infinitesimal conformal transformations.

In the same way, we can calculate the time and space components of the 'electromagnetic' four-potential A. From equations (4.14), (4.17), (4.19) and (4.20), we finally get

$$\mathscr{D}V-\boldsymbol{\phi}\,.\,\boldsymbol{A}-2t\boldsymbol{c}\,.\,\boldsymbol{A}+2c^0\boldsymbol{r}\,.\,\boldsymbol{A}-[\rho+2c\,.\,x][1+x\,.\,\nabla]V+x^2(c\,.\,\nabla)V=0 \qquad (4.22a)$$

and

$$\mathscr{D}\boldsymbol{A}+\boldsymbol{\theta}\wedge\boldsymbol{A}-V\boldsymbol{\phi}-2(x\,.\,A)\boldsymbol{c}+2(c\,.\,A)\boldsymbol{r}-[\rho+2c\,.\,x][1+x\,.\,\nabla]\boldsymbol{A}+x^2(c\,.\,\nabla)\boldsymbol{A}=0. \qquad (4.22b)$$

These equations (4.22a) and (4.22b) are the necessary and sufficient conditions of invariance of a general 'electromagnetic' four-vector under infinitesimal conformal transformations.

At this stage, the sets of equations (4.21a, b), (4.22a, b) and equation (4.15) become the fundamental relations which have to be applied in the specific cases announced in the introduction.

As a final comment, let us mention that equations (4.21a) and (4.21b) are the generalisations in the context of the conformal group of the Bacry–Combe–Richard (1970) and of the Combe–Sorba (1975) conditions on invariant electromagnetic tensors under the Poincaré group. In the same way, equations (4.22a) and (4.22b) are the generalisations of the Beckers–Comté (1979) conditions on invariant electromagnetic four-vectors under the Euclidean group in three dimensions, conditions discussed in connection with other recent contributions (Beckers *et al* 1977, Giovannini 1977).

5. Invariant tensor fields under some maximal subgroups of O(4, 2)

Let us illustrate the infinitesimal method and thus solve each set of equations (4.15), (4.21) and (4.22) in some specific cases of special interest. The three maximal subgroups of the conformal group that we want to consider are those mentioned in § 2, i.e. SO(3)⊗SO(2, 1), O(3, 2) and O(2)⊗O(4). For each of these subgroups we determine the invariant scalar density $\Phi(x)$ starting from equation (4.15). Then, in the case of O(3, 2), we show, in addition, from equations (4.21) and (4.22) that the invariant F and A fields are trivial, and for O(2)⊗O(4) we discuss from equation (4.22) the determination of the nontrivial invariant A field.

5.1. *Scalar density* Φ *invariant under* SO(3)⊗SO(2, 1)

The maximal subgroup SO(3)⊗SO(2, 1) is of order six: its Lie algebra is defined by the basis $(\boldsymbol{J}, P^0, D, C^0)$.

5.1.1. Invariance under P^0. $(a=(1,0,0,0), \boldsymbol{\theta}=\boldsymbol{\phi}=0,\ \rho=0, c^\mu=0)$ implies from (4.20) and (4.15)

$$\mathscr{D}\equiv-(a\,.\,\nabla)=-\partial/\partial t \qquad \partial\Phi/\partial t=0,$$

so that Φ is time independent: $\Phi=\Phi(\boldsymbol{r})$.

5.1.2. Invariance under D $(\rho>0,\ \boldsymbol{\theta}=\boldsymbol{\phi}=0,\ a^\mu=c^\mu=0)$ with $\mathscr{D}\equiv0$ and the time independence of Φ leads to

$$(1+\boldsymbol{r}\,.\,\partial/\partial\boldsymbol{r})\Phi(\boldsymbol{r})=0. \qquad (5.1)$$

This Euler equation indicates that Φ is a homogeneous function of dimension $d=-1$ with respect to the Cartesian coordinates. The same result is obtained directly from the invariance under C^0.

5.1.3. Rotational invariance evidently gives

$$\Phi = \Phi(|\boldsymbol{r}|) = \Phi(r) \tag{5.2}$$

so that equations (5.1) and (5.2) impose that the scalar density invariant under $SO(3)\otimes SO(2,1)$ necessarily has the form

$$\Phi = Cr^{-1} \qquad C = \text{constant}. \tag{5.3}$$

Let us end this subsection by noting that the specific determinations of an invariant F field from (4.21) and of an invariant A field from (4.22) have been effected in a different but equivalent context (Beckers *et al* 1978). The corresponding F field is nontrivial although there is no nontrivial A field invariant under $SO(3)\otimes SO(2,1)$.

5.2. Scalar density, F and A fields invariant under O(3, 2)

The maximal subgroup O(3, 2)—one of the de Sitter groups—is of order ten: its Lie algebra is defined by the basis $(\boldsymbol{J}, \boldsymbol{K}, P^\mu + C^\mu)$.

5.2.1. Scalar density Φ invariant under O(3, 2). Invariance under $P^\mu + C^\mu$ corresponds to the set of parameters

$$\begin{aligned} &a^{(\mu)} = (\delta_0^\mu, \delta_1^\mu, \delta_2^\mu, \delta_3^\mu) \qquad && c^{(\mu)} = (\delta_0^\mu, \delta_1^\mu, \delta_2^\mu, \delta_3^\mu) \\ &\boldsymbol{\theta} = \boldsymbol{\phi} = 0 && \rho = 0 \end{aligned} \tag{5.4}$$

and to the differential operator (4.20);

$$\mathscr{D}^{(\mu)} \equiv -a^{(\mu)} \,.\, \nabla = -\partial/\partial x^\mu = -\partial_\mu. \tag{5.5}$$

Then equations (4.15), (5.4) and (5.5) lead to the four relations

$$(1-x^2)\partial_\mu \Phi(x) + 2x_\mu(1+x\,.\,\nabla)\Phi(x) = 0 \qquad (\mu = 0, 1, 2, 3). \tag{5.6}$$

A little manipulation on this set gives the relation

$$(1+x^2)\partial\Phi/\partial t + 2t\Phi = 0. \tag{5.7}$$

Its general solution has the form

$$\begin{aligned} \Phi(x) &= C(\boldsymbol{r}) \exp\left(-\int \frac{2t\,\mathrm{d}t}{t^2 - r^2 + 1}\right) \\ &= C(\boldsymbol{r})/(x^2+1). \end{aligned} \tag{5.8}$$

Introducing the solution (5.8) in the original equations (5.6), we find that $C(\boldsymbol{r})$ has to be constant. Hence the scalar density invariant under O(3, 2) is

$$\Phi(x) = C/(x^2+1) \qquad C = \text{constant}. \tag{5.9}$$

It can easily be shown that invariance under the Lorentz generators $(\boldsymbol{J}, \boldsymbol{K})$ does not alter this result.

5.2.2. 'Electromagnetic' F tensor invariant under O(3, 2)

(i) *Rotational invariance* expressed through equations (4.21*a*) and (4.21*b*) corresponds to

$$\boldsymbol{E} = k(r, t)\boldsymbol{r}$$
$$\boldsymbol{B} = k'(r, t)\boldsymbol{r}. \tag{5.10}$$

(ii) *Invariance under* $P^\mu + C^\mu$ gives, with the parameters (5.4) and the differential operator (5.5), four pairs of vector equations. By simple combinations of certain of these 24 relations, it is easy to show, with (5.10), that we necessarily have

$$k = k' = 0 \tag{5.11}$$

so that there is no invariant 'electromagnetic' tensor in this case.

5.2.3. 'Electromagnetic' four-vector A invariant under O(3, 2)

(i) *Rotational invariance* gives once more

$$\boldsymbol{A} = k(r, t)\boldsymbol{r} \qquad \text{and} \qquad V = V(r, t). \tag{5.12}$$

(ii) *Invariance under* $P^\mu + C^\mu$ gives, with the parameters (5.4) and the differential operator (5.5), the four relations obtained from equations (4.22):

$$(x^2 - 1)(\partial V/\partial t) + 2\boldsymbol{r} \cdot \boldsymbol{A} - 2t(1 + x \cdot \nabla) V = 0$$

and

$$(x^2 - 1)\partial_i V + 2tA_i - 2x_i(1 + x \cdot \nabla) V = 0 \qquad (i = 1, 2, 3). \tag{5.13}$$

A little manipulation of these equations (5.13) leads to the relation

$$(x^2 + 1)(x \cdot \nabla) V + 2x^2 V = 0$$

or

$$(x \cdot \nabla) W = 0 \tag{5.14}$$

if we define

$$W = (x^2 + 1) V. \tag{5.15}$$

The Euler equations (5.14) ensures the homogeneous character of W with respect to the coordinates x^μ and its dimension $d = 0$. We conclude that W is an arbitrary constant and consequently that

$$V = C/(x^2 + 1) \qquad C = \text{constant}. \tag{5.16}$$

(iii) *Invariance under* K^x corresponds to the set of parameters

$$\boldsymbol{\phi} = (1, 0, 0) \qquad \boldsymbol{\theta} = 0 \qquad \rho = 0 \qquad a^\mu = c^\mu = 0 \tag{5.17}$$

and to the differential operator

$$\mathscr{D} \equiv x(\partial/\partial t) + t(\partial/\partial x) \tag{5.18}$$

introduced in equations (4.22). We get the four relations

$$[x(\partial/\partial t) + t(\partial/\partial x)] V - A^x = 0 \tag{5.19}$$

and

$$[x(\partial/\partial t)+t(\partial/\partial x)]\begin{pmatrix}A^x\\A^y\\A^z\end{pmatrix}-\begin{pmatrix}V\\0\\0\end{pmatrix}=0. \tag{5.20}$$

Substituting the expression (5.16) for V into (5.19), we immediately get

$$A^x=0 \tag{5.21}$$

which, introduced in the first relation (5.20), requires

$$V=0. \tag{5.22}$$

The invariance under K^y and K^z immediately gives with (5.22) that

$$A^y=A^z=0 \tag{5.23}$$

so that there exists no nontrivial invariant four-vector in the case of the O(3, 2) subgroup.

5.3. Scalar density Φ *and A field invariant under* $O(2)\otimes O(4)$

The maximal subgroup $O(2)\otimes O(4)$ is of order seven: its Lie algebra is defined by the basis ($\boldsymbol{J}$, P^0+C^0, $\boldsymbol{P}-\boldsymbol{C}$).

5.3.1. Scalar density Φ *invariant under* $O(2)\otimes O(4)$

(i) *Rotational invariance* immediately requires

$$\Phi(x)=\Phi(r,t). \tag{5.24}$$

(ii) *Invariance under* P^0+C^0 corresponds to the set of parameters

$$a=(1,0,0,0) \qquad c=(1,0,0,0) \qquad \boldsymbol{\theta}=\boldsymbol{\phi}=0 \qquad \rho=0 \tag{5.25}$$

and to the differential operator (4.20):

$$\mathscr{D}\equiv-\partial/\partial t \tag{5.26}$$

so that equation (4.15) leads to

$$(x^2-1)(\partial\Phi/\partial t)-2t(1+x\,.\,\nabla)\Phi=0. \tag{5.27}$$

(iii) *Invariance under* P^i-C^i ($i=1,2,3$) corresponds to the set of parameters

$$a^{(i)}=(0,\delta_1^i,\delta_2^i,\delta_3^i) \qquad c^{(i)}=(0,\delta_1^i,\delta_2^i,\delta_3^i) \qquad \boldsymbol{\theta}=\boldsymbol{\phi}=0 \qquad \rho=0 \tag{5.28}$$

and to the differential operator (4.20):

$$\mathscr{D}^{(i)}\equiv-a^{(i)}\,.\,\nabla=-\partial/\partial x^i=-\partial_i. \tag{5.29}$$

Equation (4.15) gives the three relations

$$(x^2+1)\partial_i\Phi-2x_i(1+x\,.\,\nabla)\Phi=0 \qquad (i=1,2,3). \tag{5.30}$$

A little manipulation on equations (5.27) and (5.30) leads to the equation

$$\frac{(1-x^2)^2+4t^2}{2t}\frac{\partial\Phi}{\partial t}+(x^2+1)\Phi=0. \tag{5.31}$$

Its general solution has the following form when equation (5.24) is taken into account:

$$\Phi(r,t)=C(r)\exp\left(-2\int\frac{(x^2+1)t\,\mathrm{d}t}{(1-x^2)^2+4t^2}\right) \tag{5.32}$$

$$=C(r)\exp\left(-\tfrac{1}{2}\ln[(1-x^2)^2+4t^2]\right)$$

$$=C'(r)[\tfrac{1}{4}(1-x^2)^2+t^2]^{-1/2}. \tag{5.33}$$

Now, if we introduce (5.33) in the original equations (5.27) and (5.30), we see that $C'(r)$ has to be constant so that the final form of the scalar density invariant under $\mathrm{O}(2)\otimes\mathrm{O}(4)$ is

$$\Phi=C'[\tfrac{1}{4}(1-x^2)^2+t^2]^{-1/2} \qquad C'=\text{constant}. \tag{5.34}$$

5.3.2. 'Electromagnetic' four-vector A invariant under $\mathrm{O}(2)\otimes\mathrm{O}(4)$

(i) *Rotational invariance* gives

$$\boldsymbol{A}(x)=k(r,t)\boldsymbol{r}=k'(r,t)\boldsymbol{r}/r \tag{5.35}$$

and

$$V(x)=V(r,t) \tag{5.36}$$

where the last equality in (5.35) is introduced for later convenience.

(ii) *Invariance under* P^0+C^0 with (5.25) and (5.26) gives in equations (4.22)

$$(1-x^2)(\partial V/\partial t)-2\boldsymbol{r}\cdot\boldsymbol{A}+2t(1+x\cdot\nabla)V=0 \tag{5.37}$$

and

$$(1-x^2)(\partial\boldsymbol{A}/\partial t)-2V\boldsymbol{r}+2t(1+x\cdot\nabla)\boldsymbol{A}=0. \tag{5.38}$$

(iii) *Invariance under* P^i-C^i $(i=1,2,3)$ with (5.28) and (5.29) leads from equations (4.22) to the set of relations

$$(1+x^2)\partial_i V+2tA_i-2x_i(1+x\cdot\nabla)V=0 \tag{5.39}$$

and

$$(1+x^2)\partial_i\boldsymbol{A}-2(x\cdot\boldsymbol{A})\boldsymbol{e}_i+2A_i\boldsymbol{r}-2x_i(1+x\cdot\nabla)\boldsymbol{A}=0 \tag{5.40}$$

where

$$\boldsymbol{e}_i=(\delta_i^1,\delta_i^2,\delta_i^3) \qquad (i=1,2,3).$$

The integration of the system of equations (5.37)–(5.40) is somewhat tedious and we give more details on the explicit calculations in the appendix. Here let us mention that a little manipulation on equations (5.37) and (5.39) leads to the following relation between V and its derivatives:

$$(1-x^2)(x\cdot\nabla)V-2x^2V-2t\partial V/\partial t=0. \tag{5.41}$$

Its general solution may be written (cf appendix)

$$V(r,t)=\mathscr{A}(\xi)/[1+(t+r)^2] \tag{5.42}$$

where $\mathscr{A}(\xi)$ is an arbitrary function of the variable ξ defined by

$$\xi = (1 + (t-r)^2]/[1 + (t+r)^2]. \tag{5.43}$$

Similar transformations on equations (5.38) and (5.40) permit us to obtain a relation between $\boldsymbol{A}$ and its derivatives:

$$(1 - x^2)(x \,.\, \nabla)\boldsymbol{A} - 2x^2\boldsymbol{A} - 2t\partial\boldsymbol{A}/\partial t = 0. \tag{5.44}$$

From (5.35) and (5.44), we can see that $k'(r, t)$ satisfies equation (5.41), like V, and may thus be written

$$k'(r, t) = \mathscr{B}(\xi)/[1 + (t+r)^2] \tag{5.45}$$

where the form of $\mathscr{B}(\xi)$ is still to be determined.

In order to precise the functions $\mathscr{A}(\xi)$ and $\mathscr{B}(\xi)$ we have to consider again the original equations (5.37)–(5.40). In fact, it is shown in the appendix that equation (5.37) is equivalent to the following relation when equations (5.35), (5.42) and (5.45) are used:

$$\mathscr{A}(\xi) + \mathscr{B}(\xi) + 2\xi\partial\mathscr{A}(\xi)/\partial\xi = 0 \tag{5.46}$$

and that equation (5.38) is equivalent to

$$\mathscr{B}(\xi) + \mathscr{A}(\xi) + 2\xi\partial\mathscr{B}(\xi)/\partial\xi = 0. \tag{5.47}$$

The solutions of (5.46) and (5.47) are given by

$$\mathscr{A}(\xi) = (K_1/\xi) + K_2 \tag{5.48}$$

and

$$\mathscr{B}(\xi) = (K_1/\xi) - K_2 \tag{5.49}$$

where K_1 and K_2 are arbitrary constants. As a last step, we prove in the appendix that

$$K_1 = K_2 = K. \tag{5.50}$$

This result appears as a consequence of the original equation (5.40) when (5.35), (5.42), (5.45), (5.48) and (5.49) are taken into account.

Finally, collecting all these partial results (equations (5.35), (5.36), (5.42), (5.43), (5.45), (5.48), (5.49) and (5.50)), we find the explicit solutions

$$\boldsymbol{A} = \frac{K(\xi^{-1} - 1)}{1 + (t+r)^2}\,\frac{\boldsymbol{r}}{r} = C\frac{t\boldsymbol{r}}{t^2 + \frac{1}{4}(1 - x^2)^2} \tag{5.51}$$

$$V = \frac{K(\xi^{-1} + 1)}{1 + (t+r)^2} = C'\frac{1 + t^2 + r^2}{t^2 + \frac{1}{4}(1 - x^2)^2} \tag{5.52}$$

where C and C' are arbitrary constants. These forms are the 'electromagnetic' potentials invariant under $\mathrm{O}(2)\otimes\mathrm{O}(4)$.

For the sake of completeness, let us mention that it can be shown very easily from equations (4.21) that there is no invariant F field in the case of $\mathrm{O}(2)\otimes\mathrm{O}(4)$; the demonstration is completely parallel to that of § 5.2.2.

6. Discussion and conclusions

The results of § 5 can be summarised by the following conclusions:

$$
\begin{aligned}
\mathrm{SO}(3)\otimes\mathrm{SO}(2,1)&: \boldsymbol{F}\equiv(\boldsymbol{E},\boldsymbol{B})\neq 0 \qquad \text{(cf Beckers } et\ al \text{ 1978)}\\
&\quad A\equiv(V,\boldsymbol{A})=0 \qquad \text{(cf Beckers } et\ al \text{ 1978)}\\
&\quad \Phi\equiv(5.3)\neq 0,\\
\mathrm{O}(3,2)&: F=0 \qquad A=0 \qquad \Phi\equiv(5.9)\neq 0,\\
\mathrm{O}(2)\otimes\mathrm{O}(4)&: F=0 \qquad V\equiv(5.52)\neq 0 \qquad \boldsymbol{A}\equiv(5.51)\neq 0\\
&\quad \Phi\equiv(5.34)\neq 0
\end{aligned}
\tag{6.1}
$$

where all these solutions have been obtained from some of the relations constituting the systems (4.15), (4.21) and (4.22). Conversely, it is easy to verify that they satisfy *all* the equations of these systems.

The interest of these three maximal subgroups of the conformal group was mentioned in the introduction. We now have precise information on their F-, A- and Φ-invariant fields as well as on their corresponding metric tensors (Beckers and Jaminon 1978). We want to recall that all these results have also been obtained through a *global* method by Beckers *et al* (1978), who have studied all the conjugacy classes (under the Poincaré group) of the maximal subalgebras of the conformal algebra.

The specific interest of this paper consists of the exploitation of the elements discussed by FRW on conformal *point* transformations applied through our conditions (3.2), (3.5) and (3.6), and their corresponding infinitesimal forms (4.8) or (4.21), (4.14) or (4.22) and (4.15). These are effectively differential equations describing the local invariance of F, A and Φ fields under infinitesimal conformal point transformations.

Among the specific results we have obtained, let us pick out the scalar density $\Phi\equiv(5{\cdot}9)$ invariant under O(3, 2) which is exactly the expression given by Fubini (1976) as a solution of the conformal invariant field equation of the $\lambda\phi^4$ field theory. Let us also notice the fact that some nontrivial results obtained above can be generalised as follows:

$$\Phi_a = D(a^2+t^2-r^2)^{-1} \tag{6.2}$$

in the case of O(3, 2) invariance,

$$\Phi_a = D'[\tfrac{1}{4}(a^2-t^2+r^2)^2+t^2]^{-1/2} \tag{6.3}$$

$$V_a = C\frac{a^2+t^2+r^2}{\frac{1}{4}(a^2-t^2+r^2)^2+t^2} \tag{6.4}$$

and

$$\boldsymbol{A}_a = C'\frac{t\boldsymbol{r}}{\frac{1}{4}(a^2-t^2+r^2)+t^2} \tag{6.5}$$

in the case of O(2)⊗O(4) invariance. These expressions (6.2)–(6.5) contain the parameter a when we consider, as Fubini did, the operators

$$R^\mu = \tfrac{1}{2}(aP^\mu + a^{-1}C^\mu) \qquad (\mu=0,1,2,3) \tag{6.6}$$

which do introduce a fundamental length (Fubini 1976) *or* when more general coordinates are introduced (Beckers *et al* 1978).

Finally, let us add some comments on differential geometry (Hicks 1971) where the determination of tensor fields invariant under some transformation Lie groups is intimately connected with Lie derivatives. It is well known that invariance of such tensor fields corresponds to the vanishing of the Lie derivative $\mathscr{L}_\chi \Upsilon$ of these tensors Υ with respect to the vector fields χ induced by the one-parameter subgroups:

$$\mathscr{L}_\chi \Upsilon(x) \equiv \bar{\Upsilon}(x) - \Upsilon(x) = 0 \tag{6.7}$$

where

$$\chi \equiv \xi^\mu \partial_\mu \tag{6.8}$$

is the vector field associated here with the infinitesimal conformal transformations defined by (4.2) or (4.3). Applied to a skew-symmetric F tensor of degree two (two-form), to a four-vector A (one-form) and to a scalar density Φ (of dimension -1), the corresponding equations (6.7) are respectively

$$F_{\mu\nu,\alpha}\xi^\alpha + F_{\mu\alpha}\xi^\alpha{}_{,\nu} + F_{\alpha\nu}\xi^\alpha{}_{,\mu} = 0 \tag{6.9}$$

$$A_{\mu,\alpha}\xi^\alpha + A_\alpha \xi^\alpha{}_{,\mu} = 0 \tag{6.10}$$

and

$$\xi^\mu \Phi_{,\mu} + \tfrac{1}{4}\xi^\mu{}_{,\mu}\Phi = 0. \tag{6.11}$$

These conditions describe *local* invariance of the corresponding quantities under infinitesimal conformal transformations. They are equivalent to our equations (3.2), (3.5) and (3.6) established for contravariant components. This geometrical language has been used extensively in the work of Beckers *et al* (1978) and we refer the reader to that paper for further developments.

Acknowledgments

One of the authors (JB) is indebted to P Winternitz and his collaborators for interesting discussions.

Appendix

In order to justify the results (5.42), (5.45), (5.46), (5.47) and (5.50) in the determination of the four-vector A invariant under $O(2)\otimes O(4)$, let us introduce adequate variables in terms of r and t:

$$(t, r) \rightarrow (p, q): p = t + r, q = t - r \tag{A.1}$$

and let us define

$$u = 1 + p^2 \qquad v = 1 + q^2, \tag{A.2}$$

so that we immediately get the simple relations

$$\tfrac{1}{2}(u + v) = 1 + t^2 + r^2 \qquad \tfrac{1}{2}(u - v) = 2rt \tag{A.3}$$

$$\frac{v}{u} = \xi = \frac{1 + (t - r)^2}{1 + (t + r)^2} \equiv (5{\cdot}43). \tag{A.4}$$

We have to solve equation (5.41) when V is given by (5.36), i.e.

$$(1-t^2+r^2)\left(t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\right)V(r,t)-2(t^2-r^2)V(r,t)-2t\frac{\partial V}{\partial t}(r,t)=0$$

or

$$(1-t^2+r^2)r\frac{\partial V}{\partial r}-(1+t^2-r^2)t\frac{\partial V}{\partial t}-2(t^2-r^2)V=0. \qquad (A.5)$$

The solution of such an equation can be found by different methods (Sneddon 1957), but let us notice that, with (A.1), we get

$$q(p^2+1)\frac{\partial V}{\partial p}+p(q^2+1)\frac{\partial V}{\partial q}+2pqV=0. \qquad (A.6)$$

If we put

$$V=\frac{Z}{(1+p^2)(1+q^2)}=\frac{Z}{uv} \qquad (A.7)$$

we obtain the equation

$$u\frac{\partial Z}{\partial u}+v\frac{\partial Z}{\partial v}=Z \qquad (A.8)$$

showing that Z is a homogeneous function of dimension $d=1$ in u and v. Hence, we may write

$$Z=v\mathscr{A}(v/u)=v\mathscr{A}(\xi) \qquad (A.9)$$

where $\mathscr{A}(\xi)$ is an arbitrary function of $\xi\equiv(A.4)$. So we get

$$V=\frac{v\mathscr{A}(\xi)}{uv}=\frac{\mathscr{A}(\xi)}{u}\equiv(5{\cdot}42). \qquad (A.10)$$

Consequently, the result (5.45) is also established because it is very easy to show that $k'(r,t)$ defined in equation (5.35) satisfies, through (5.44), the same equation (5.41) ≡ (A.5) as V.

Now, let us prove equation (5.46). When (5.35) and (5.36) are taken into account, the relation (5.37) becomes

$$(1+t^2+r^2)(\partial V/\partial t)+2tV+2rt(\partial V/\partial r)-2rk'=0. \qquad (A.11)$$

With the choices (A.1)–(A.4) and the result (5.42), we first translate the partial derivatives of V in terms of $\mathscr{A}$ and $\mathrm{d}\mathscr{A}/\mathrm{d}\xi$. Then, after a little manipulation, we get, through (A.11) and (5.45), the announced equation

$$\mathscr{A}(\xi)+\mathscr{B}(\xi)+2\xi\,\mathrm{d}\mathscr{A}(\xi)/\mathrm{d}\xi=0. \qquad (A.12)$$

For completeness, let us notice that equations (5.39) are redundant in the sense that they also lead to equation (A.12). This is easily verified if we note that these three relations can be rewritten in vector notation in the form

$$(1+x^2)\,\mathrm{grad}\,V-2t\boldsymbol{A}+2\boldsymbol{r}(1+x\,.\,\nabla)V=0$$

so that, with (5.35), we find

$$(1+t^2+r^2)\frac{\partial V}{\partial r}+2rV+2rt\frac{\partial V}{\partial t}-2tk'=0. \tag{A.13}$$

This equation is identical with (A.11) under the interchange of the variables r and t. Then, at the level of the variables u, v or ξ, it leads to the same condition (A.12).

The proof of equation (5.47) proceeds as follows. Starting from equation (5.38), it is sufficient to introduce (5.35) and to notice that the resulting equation is identical to (A.11) under the interchange of $V(r, t)$ and $k'(r, t)$. Thus, considerations similar to those leading to (A.12) give here the relation

$$\mathscr{B}(\xi)+\mathscr{A}(\xi)+2\xi\, \mathrm{d}\mathscr{B}(\xi)/\mathrm{d}\xi=0. \tag{A.14}$$

The last point which has to be established in this appendix corresponds to equation (5.50). It can be obtained from equations (5.40) by different ways. For instance, we can use one of the equations (5.40) not containing V. So, taking $i=1$ and the y component of (5.40), we get

$$(1+t^2-r^2)\frac{\partial A^y}{\partial x}-2yA^x+2x(1+x\,.\,\nabla)A^y=0. \tag{A.15}$$

Then, with equations (5.35), (5.45), (5.48) and (5.49) introduced in (A.15), simple but lengthy calculations lead to the result

$$K_1=K_2(\equiv(5.50)). \tag{A.16}$$

Another way is to combine different components among the nine equations corresponding to (5.40) in order to get a new relation between the functions $\mathscr{A}(\xi)$ and $\mathscr{B}(\xi)$. For instance, if we take from equation (5.40) the x component for $i=1$, the y component for $i=2$ and the z component for $i=3$, we can obtain the following equation when (5.42) and (5.45) are taken into account:

$$3\mathscr{A}(\xi)+\frac{1+3\xi}{\xi-1}\mathscr{B}(\xi)+2\xi\frac{\mathrm{d}\mathscr{B}(\xi)}{\mathrm{d}\xi}=0. \tag{A.17}$$

then, with (5.48) and (5.49), we immediately get the result (A.16).

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